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Recurrence relations for radial wavefunctions for the *N*th-dimensional oscillators and hydrogenlike atoms

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Abstract
We present a general procedure for finding recurrence relations of the radial wavefunctions for *N*th-dimensional isotropic harmonic oscillators and hydrogenlike atoms.

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1. Introduction

There are many applications in modern physics that require knowledge of the wavefunctions of hydrogenlike atoms and isotropic harmonic oscillators, especially for finding the corresponding matrix elements (see, e.g., [18, 23] and references therein). There are several methods for generating such wavefunctions among which the so-called factorization method of Infeld and Hull [12] is of particular significance (for more recent papers see, e.g., [3, 13, 17, 24]). Moreover, the recurrence relations and the *ladder-type* operators for these wavefunctions are useful for finding the transition probabilities and evaluation of certain integrals [18, 23]. Methods for obtaining such recurrence relations have attracted the interest of several authors (see, e.g., [6, 21, 22]), and usually are based on the connection of such functions with the classical Laguerre polynomials. For generating further 'non-trivial' relations a Laplace-transform-based method has been developed recently [6, 25] but the calculation is cumbersome and requires inversion formulae.

Our main aim in this paper is to present an alternative approach for generating recurrence relations and ladder-type operators for the *N*th-dimensional isotropic harmonic oscillators and hydrogenlike atoms. The idea is to exploit the connection of the radial wavefunctions with the classical Laguerre polynomials and use a general theorem for the hypergeometric-type functions [20] that will allow us to obtain several new relations for these polynomials and therefore, for the wavefunctions. Its advantage, compared to the other approaches, is not only that it can be easily extended to other exactly solvable models which involve hypergeometric functions or polynomials (see, e.g., [5]), but also because it is a constructive method: one chooses the values of the parameters and one gets the corresponding coefficients. The same method has been recently used [7, 9, 10, 27, 28] to derive new recurrence relations for the hypergeometric polynomials and functions, i.e. the solutions of the second-order differential equation $\sigma(x)y'' + \tau(x)y' + \lambda y = 0$, being σ and τ polynomials of degree, at most, 2 and 1, respectively. We will only show here how this method works in some representative examples and refer to [7] for a more detailed list.

The structure of the paper is as follows: In section 2 the needed results and notation from the special function theory are introduced. In section 3 the isotropic oscillator is introduced and several recurrence and ladder-type relations are obtained. Similar results for the hydrogenlike atoms are presented in section 4. Finally, relevant references are quoted.

2. Preliminaries

In this paper we will deal with the hypergeometric functions y_ν, which are the solution of the hypergeometric-type differential equation

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0 \tag{2.1}$$

where σ and τ are polynomials with $\deg\sigma \leqslant 2$, $\deg\tau \leqslant 1$ and λ is a constant. The solutions of this equation are usually called functions of hypergeometric type and are denoted by $y_\nu(z)$ where ν is such that $\lambda = -\nu\tau' - \nu(\nu-1)\sigma''/2$. The functions y_ν have the form [20]

$$y(z) = y_\nu(z) = \frac{C_\nu}{\rho(z)} \int_C \frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu+1}}\,\mathrm{d}s \tag{2.2}$$

where ρ is a solution of the Pearson equation $(\sigma\rho)' = \tau\rho$, σ and τ do not depend on ν, C is a contour in the complex plane such that its end points s_1 and s_2 satisfy the condition

$$\left.\frac{\sigma^{\nu+1}(s)\rho(s)}{(s-z)^{\nu+2}}\right|_{s_1}^{s_2} = 0 \tag{2.3}$$

and C_ν is a normalizing factor.

For the hypergeometric functions y_ν the following theorem holds [20, page 18]:

Theorem 2.1. *Let $y_{\nu_i}^{(k_i)}(z)$, $i = 1, 2, 3$, be any three derivatives of order k_i of the functions of hypergeometric type, $\nu_i - \nu_j$ being an integer and such that*

$$\left.\frac{\sigma^{\nu_0+1}(s)\rho(s)}{(s-z)^{\mu_0}}s^m\right|_{s_1}^{s_2} = 0 \qquad m = 0, 1, 2, \ldots$$

where ν_0 denotes the index ν_i with minimal real part and μ_0 that with maximal real part. Then, there exist three non-vanishing polynomials $B_i(z)$, $i = 1, 2, 3$, such that

$$\sum_{i=1}^{3} B_i(z) y_{\nu_i}^{(k_i)}(z) = 0. \tag{2.4}$$

A special case of these functions is the polynomials of hypergeometric type, i.e. the polynomial solutions of equation (2.1). They are defined by [20]

$$p_n(z) = \frac{C_n}{\rho(z)} \oint \frac{\sigma^n(s)\rho(s)}{(s-z)^{n+1}}\,\mathrm{d}s \qquad n = 0, 1, 2, \ldots$$

i.e., the same function y_ν of the expression (2.2) but the contour C is closed and ν is a non-negative integer. Note that, in this case, the condition (2.3) is automatically fulfilled, so the theorem 2.1 holds for any family of polynomials of hypergeometric type. Note also that theorem 2.1 assures the existence of the non-vanishing polynomials in (2.4) but does not give any method for finding them. In general, it is not easy to find explicit expressions for these polynomials B_i but, in some cases [7, 9–11, 27, 28], these coefficients are obtained explicitly in terms of the coefficients of the polynomials σ and τ in (2.1).

An example of such polynomials is the Laguerre polynomials L_n^α defined by the hypergeometric series

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\,\middle|\, x\right) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}$$
$$(a)_0 := 1 \qquad (a)_k := a(a+1)(a+2)\cdots(a+k-1) \qquad k = 1, 2, 3, \ldots. \tag{2.5}$$

These polynomials satisfy the following recurrence and differential–recurrence relations useful in the following sections (see, e.g., [1, 20, 26])

$$\frac{\mathrm{d}}{\mathrm{d}x}L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) \tag{2.6}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}L_n^\alpha(x) = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x) = (n+1)L_{n+1}^\alpha(x) - (n+\alpha+1-x)L_n^\alpha(x) \tag{2.7}$$

$$xL_n^{\alpha+1}(x) = (n+\alpha+1)L_n^\alpha(x) - (n+1)L_{n+1}^\alpha(x) \tag{2.8}$$

$$xL_n^{\alpha+1}(x) = (n+\alpha)L_{n-1}^\alpha(x) - (n-x)L_n^\alpha(x) \tag{2.9}$$

$$L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x) \tag{2.10}$$

$$(n+1)L_{n+1}^\alpha(x) - (2n+\alpha+1-x)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0. \tag{2.11}$$

Other instances of hypergeometric polynomials are the Jacobi, Bessel and Hermite polynomials [8, 20, 26].

3. The isotropic harmonic oscillator

The N-dimensional isotropic harmonic oscillator (IHO) is described by the Schrödinger equation

$$\left(-\Delta + \frac{1}{2}\lambda^2 r^2\right)\Psi = E\Psi \qquad \Delta = \sum_{k=1}^{n}\frac{\partial}{\partial x_k} \qquad r = \sqrt{\sum_{k=1}^{n} x_k^2}.$$

For solving it one uses the method of separation of variables that leads to a solution of the form $\Psi = R_{nl}^{(N)}(r)Y_{lm}(\Omega_N)$, where $R_{nl}^{(N)}(r)$ is the radial part, usually called the *radial wavefunctions*, defined by (see, e.g., [4, 6])

$$R_{nl}^{(N)}(r) = \mathcal{N}_{nl}^{(N)} r^l\, \mathrm{e}^{-\frac{1}{2}\lambda r^2} L_n^{l+\frac{N}{2}-1}(\lambda r^2) \qquad \mathcal{N}_{nl}^{(N)} = \sqrt{\frac{2n!\lambda^{l+\frac{N}{2}}}{\Gamma\left(n+l+\frac{N}{2}\right)}} \tag{3.1}$$

$n = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$ being the quantum numbers, and $N \geqslant 3$ the dimension of the space. The angular part $Y_{lm}(\Omega_N)$ is the so-called Nth-spherical or hyperspherical harmonics [4, 19]. In the following, we will assume that the parameters n, l, N are non-negative integers.

3.1. Recurrence relations connecting three different radial functions

We will look, now, for recurrence relations connecting three different radial functions.

Theorem 3.1. *Let $R_{nl}^{(N)}(r)$, $R_{n+n_1,l+l_1}^{(N)}(r)$ and $R_{n+n_2,l+l_2}^{(N)}(r)$ be three different radial functions of the Nth-dimensional isotropic harmonic oscillator, where n_1, n_2 and l_1, l_2 are integers such that $\min(n+n_1, n+n_2, l+l_1, l+l_2) \geqslant 0$. Then, there exist non-vanishing polynomials in r, A_0, A_1, and A_2, such that*

$$A_0 R_{n,l}^{(N)}(r) + A_1 R_{n+n_1,l+l_1}^{(N)}(r) + A_2 R_{n+n_2,l+l_2}^{(N)}(r) = 0. \tag{3.2}$$

Proof. For the sake of simplicity we will prove the theorem for the case when l_1, l_2 are nonnegative integers. The case when l_1, l_2 are *integers* can be derived in the same way since the 'cases' $R_{nl}^{(N)}$, $R_{n+n_1,l\pm l_1}^{(N)}$, $R_{n+n_2,l\pm l_2}^{(N)}$ can be reduced to $R_{nl}^{(N)}$, $R_{n+n_1,l+l_1}^{(N)}$, $R_{n+n_2,l+l_2}^{(N)}$ by choosing $l = \min(l, l \pm l_1, l \pm l_2)$.

From (3.1) we have

$$L_n^{l+\frac{N}{2}-1}(\lambda r^2) = \left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} r^{-l}\, \mathrm{e}^{\frac{1}{2}\lambda r^2} R_{nl}^{(N)}(r). \tag{3.3}$$

Thus,

$$L_{n+n_1}^{(l+l_1)+\frac{N}{2}-1}(\lambda r^2) = \left(\mathcal{N}_{n+n_1,l+l_1}^{(N)}\right)^{-1} r^{-(l+l_1)}\, \mathrm{e}^{\frac{1}{2}\lambda r^2} R_{n+n_1,l+l_1}^{(N)}(r) \tag{3.4}$$

$$L_{n+n_2}^{(l+l_2)+\frac{N}{2}-1}(\lambda r^2) = \left(\mathcal{N}_{n+n_2,l+l_2}^{(N)}\right)^{-1} r^{-(l+l_2)}\, \mathrm{e}^{\frac{1}{2}\lambda r^2} R_{n+n_2,l+l_2}^{(N)}(r). \tag{3.5}$$

Putting $s = \lambda r^2$, it is possible to rewrite the left-hand side of (3.4) and (3.5) in the form

$$L_{n+n_1}^{(l+l_1)+\frac{N}{2}-1}(s) = (-1)^{l_1} \frac{\mathrm{d}^{l_1}}{\mathrm{d}s^{l_1}} L_{n+n_1+l_1}^{l+\frac{N}{2}-1}(s) \qquad L_{n+n_2}^{(l+l_2)+\frac{N}{2}-1}(s) = (-1)^{l_2} \frac{\mathrm{d}^{l_2}}{\mathrm{d}s^{l_2}} L_{n+n_2+l_2}^{l+\frac{N}{2}-1}(s) \tag{3.6}$$

respectively. On the other hand, by the generalized three-term recurrence relation (2.4), there exist non-vanishing polynomials $B_i(s)$, $i = 0, 1, 2$, such that

$$B_0 L_n^{l+\frac{N}{2}-1}(s) + B_1 \frac{\mathrm{d}^{l_2}}{\mathrm{d}s^{l_2}} L_{n+n_1+l_1}^{l+\frac{N}{2}-1}(s) + B_2 \frac{\mathrm{d}^{l_1}}{\mathrm{d}s^{l_1}} L_{n+n_2+l_2}^{l+\frac{N}{2}-1}(s) = 0.$$

Thus, since (3.6),

$$C_0 L_n^{l+\frac{N}{2}-1}(s) + C_1 L_{n+n_1}^{(l+l_1)+\frac{N}{2}-1}(s) + C_2 L_{n+n_2}^{(l+l_2)+\frac{N}{2}-1}(s) = 0 \tag{3.7}$$

where $C_0 = B_0$, $C_1 = (-1)^{l_1} B_1$ and $C_2 = (-1)^{l_2} B_2$. If we substitute (3.3)–(3.5) in (3.7), we have

$$\left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} C_0 R_{nl}^{(N)}(r) + \left(\mathcal{N}_{n+n_1,l+l_1}^{(N)}\right)^{-1} C_1 r^{-l_1} R_{n+n_1,l+l_1}^{(N)}(r) + \left(\mathcal{N}_{n+n_2,l+l_2}^{(N)}\right)^{-1} C_2 r^{-l_2} R_{n+n_2,l+l_2}^{(N)}(r) = 0 \tag{3.8}$$

which transforms into (3.2) where $A_0 = \left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} C_0 r^{l_1+l_2}$, $A_1 = \left(\mathcal{N}_{n+n_1,l+l_1}^{(N)}\right)^{-1} C_1 r^{l_2}$ and $A_2 = \left(\mathcal{N}_{n+n_2,l+l_2}^{(N)}\right)^{-1} C_2 r^{l_1}$. Obviously these functions A_i, $i = 0, 1, 2$, are polynomials in r. □

Using this technique, it is very simple to obtain concrete relations between three different *radial functions* of the IHO. Here we will present five of them. As we already pointed out, it is not easy to obtain the coefficients C_i in (3.7), nevertheless, combining in a certain way the properties (2.6)–(2.11) they can be easily identified. In the following examples we can show how the coefficients C_i can be 'guessed'. The idea is as follows: one should first decide the relation to be obtained by defining the values n_1, n_2, l_1 and l_2 in (3.7), and then, combining in a certain way equations (2.6)–(2.11), transform (3.7) into one of the formulae (2.6)–(2.11) or in a sum of linearly independent Laguerre polynomials from where the unknown coefficients easily follow. Let us show how this works.

- Substituting $n_1 = -1, n_2 = 1, l_1 = l_2 = 0$ and $\alpha = l + \frac{N}{2} - 1$ in (3.7) it becomes

$$C_0 L_n^{\alpha}(s) + C_1 L_{n-1}^{\alpha}(s) + C_2 L_{n+1}^{\alpha}(s) = 0.$$

Comparing the above expression with the relation (2.11) we see that the coefficients $C_i, i = 1, 2, 3$ are given by $C_0 = s - (2n + \alpha + 1), C_1 = n + \alpha$ and $C_2 = n + 1$. Substituting these values in (3.8) we find the following recurrence relation

$$\sqrt{n\left(n + l + \frac{N}{2} - 1\right)} R_{n-1,l}^{(N)}(r) + \left[\lambda r^2 - \left(2n + l + \frac{N}{2}\right)\right] R_{n,l}^{(N)}(r) + \sqrt{(n+1)\left(n + l + \frac{N}{2}\right)} R_{n+1,l}^{(N)}(r) = 0.$$

This relation generalizes the recurrence relation obtained in [13] for the Laguerre functions.

The next four formulae can be obtained in an analogous way. We will only indicate here the values of the integers n_1, n_2, l_1 and l_2, respectively.

- $n_1 = n_2 = 0, l_1 = -1, l_2 = 1$:

$$r\sqrt{\lambda\left(n + l + \frac{N}{2} - 1\right)} R_{n,l-1}^{(N)}(r) - \left(l + \frac{N}{2} - 1 + \lambda r^2\right) R_{n,l}^{(N)}(r) + r\sqrt{\lambda\left(n + l + \frac{N}{2}\right)} R_{n,l+1}^{(N)}(r) = 0.$$

- $n_1 = 0, n_2 = 1, l_1 = -1, l_2 = 0$:

$$\sqrt{\lambda\left(n + l + \frac{N}{2} - 1\right)} R_{n,l-1}^{(N)}(r) + (n + 1 - \lambda r^2) R_{n,l}^{(N)}(r) - \sqrt{(n+1)\left(n + l + \frac{N}{2}\right)} R_{n+1,l}^{(N)}(r) = 0.$$

- $n_1 = -1, n_2 = 0, l_1 = 0, l_2 = 1$:

$$-\sqrt{n\left(n + l + \frac{N}{2} - 1\right)} R_{n-1,l}^{(N)}(r) + (n - \lambda r^2) R_{n,l}^{(N)}(r) + r\sqrt{\lambda\left(n + l + \frac{N}{2}\right)} R_{n,l+1}^{(N)}(r) = 0.$$

- $n_1 = 0, n_2 = 1, l_1 = -1, l_2 = 0$:

$$r\sqrt{\lambda\left(n + l + \frac{N}{2} - 1\right)} R_{n,l-1}^{(N)}(r) + (n + 1 - \lambda r^2) R_{n,l}^{(N)}(r) - \sqrt{(n+1)\left(n + l + \frac{N}{2}\right)} R_{n+1,l}^{(N)}(r) = 0.$$

Obviously, similar recurrence relations for the radial functions $R_{n,l}^{(N)}(r)$ can be found using the same procedure, i.e. choosing different values for the parameters n_1, n_2, l_1 and l_2.

3.2. Ladder-type relations for the IHO radial functions

Next we will establish a linear relation involving two radial functions of the IHO and the derivative of one of them. Some of these relations will define the so-called ladder operators for the radial wavefunctions and have important applications in the so-called factorization method (see, e.g., [3, 12, 13, 18, 24]).

Theorem 3.2. *Let $R_{n,l}^{(N)}(r)$ and $R_{n+n_1,l+l_1}^{(N)}(r)$ be two radial functions of the Nth-dimensional isotropic harmonic oscillator and let* $\min(n+n_1, l+l_1) \geqslant 0$ *and* $(n_1)^2 + (l_1)^2 \neq 0$, *where n_1 and l_1 are integers. Then, there exist non-vanishing polynomials in r, A_0, A_1, and A_2, such that*

$$A_0 R_{n,l}^{(N)}(r) + A_1 \frac{\mathrm{d}}{\mathrm{d}r} R_{n,l}^{(N)}(r) + A_2 R_{n+n_1,l+l_1}^{(N)}(r) = 0. \tag{3.9}$$

Proof. Again we will give the proof for the case when l_1 is a non-negative integer. The case when l_1 is an *integer* can be derived in the same way by choosing $l = \min(l, l \pm l_1)$. We start by taking the derivative of (3.3)

$$\left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} \left[\frac{\mathrm{d}\left(r^{-l}\,\mathrm{e}^{\frac{1}{2}\lambda r^2}\right)}{\mathrm{d}r} R_{nl}^{(N)}(r) + r^{-l}\,\mathrm{e}^{\frac{1}{2}\lambda r^2} \frac{\mathrm{d}}{\mathrm{d}r} R_{nl}^{(N)}(r) \right] = \frac{\mathrm{d}}{\mathrm{d}r} L_n^{l+\frac{N}{2}-1}(\lambda r^2)$$
$$= 2\lambda r \frac{\mathrm{d}}{\mathrm{d}s} L_n^{l+\frac{N}{2}-1}(s) \tag{3.10}$$

where $s = \lambda r^2$. Using the last expression and the identities

$$L_{n-1}^{l+\frac{N}{2}}(s) = -\left(L_n^{l+\frac{N}{2}-1}(s)\right)' \qquad \text{and} \qquad L_{n+n_1}^{(l+l_1)+\frac{N}{2}-1}(s) = (-1)^{l_1} \frac{\mathrm{d}^{l_1}}{\mathrm{d}s^{l_1}} L_{n+n_1+l_1}^{l+\frac{N}{2}-1}(s)$$

as well as theorem 2.1 we can guarantee that there exists a recurrence relation with non-vanishing polynomial coefficients B_0, B_1 and B_2

$$B_0 L_n^{l+\frac{N}{2}-1}(s) + B_1 L_{n-1}^{l+\frac{N}{2}}(s) + B_2 L_{n+n_1}^{(l+l_1)+\frac{N}{2}-1}(s) = 0. \tag{3.11}$$

Hence, from (3.10), (3.11) and (3.3)

$$\left[B_1 \frac{\mathrm{d}}{\mathrm{d}r} + \lambda r (B_1 - 2B_0) - B_1 \frac{l}{r} \right] R_{nl}^{(N)}(r) = 2\lambda B_2 \frac{\mathcal{N}_{n,l}^{(N)}}{\mathcal{N}_{n+n_1,l+l_1}^{(N)}} r^{1-l_1} R_{n+n_1,l+l_1}^{(N)}(r) \tag{3.12}$$

which completes the proof. □

Let us now show how one can obtain ladder-type relations for the radial wavefunction $R_{n,l}^{(N)}(r)$ of the IHO. Again the theorem does not give any method for getting ladder-type operators but we can proceed as follows: first, to decide the relation that one wants to obtain by defining the values n_1 and l_1 and substitute them in (3.11). Then, combining in a certain way equations (2.6)–(2.11) transform (3.11) into one of the formulae (2.6)–(2.11) or in a sum of linearly independent Laguerre polynomials and solve the resulting equations for the unknown coefficients. Let us consider some examples.

- Choosing $n_1 = -1$ and $l_1 = 1$, (3.11) becomes

  $$B_0 L_n^{\alpha}(s) + B_1 L_{n-1}^{\alpha+1}(s) + B_2 L_{n-1}^{(\alpha+1)}(s) = 0.$$

  So, using the expression (2.6)

  $$B_0 L_n^{\alpha}(s) - (B_1 + B_2)(L_n^{\alpha})'(s) = 0.$$

Since $L_n^\alpha(s)$ and $(L_n^\alpha)'(s)$ are linearly independent, $B_0 = 0$ and $B_1 = -B_2$. Then equation (3.12), with the choice $B_2 = 1$, gives

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} + \lambda r - \frac{l}{r}\right] R_{n,l}^{(N)}(r) = -2\frac{\mathcal{N}_{n,l}^{(N)}}{\mathcal{N}_{n-1,l+1}^{(N)}} R_{n-1,l+1}^{(N)}(r)$$

from where we find

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} + \lambda r - \frac{l}{r}\right] R_{n,l}^{(N)}(r) = -2\sqrt{\lambda n}\, R_{n-1,l+1}^{(N)}(r). \tag{3.13}$$

In the same way we have the following results:

- $n_1 = 1, l_1 = -1$:

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} - \lambda r + \frac{l+N-2}{r}\right] R_{n,l}^{(N)}(r) = 2\sqrt{\lambda(n+1)}\, R_{n+1,l-1}^{(N)}(r). \tag{3.14}$$

- $n_1 = 1, l_1 = 1$:

$$\begin{aligned}&\left[(\lambda r^2 - (n+1))\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l}{r} - \lambda r\right) + 2\lambda\left(n + l + \frac{N}{2}\right) r\right] R_{n,l}^{(N)}(r)\\ &\qquad = 2\sqrt{\lambda(n+1)\left(n+l+\frac{N}{2}\right)\left(n+l+\frac{N}{2}+1\right)}\, R_{n+1,l+1}^{(N)}(r).\end{aligned} \tag{3.15}$$

- $n_1 = -1, l_1 = -1$:

$$\begin{aligned}&\left[(\lambda r^2 - n)\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l}{r} + \lambda r\right) - 2n\left(\lambda r + \frac{l + \frac{N}{2} - 1}{r}\right)\right] R_{n,l}^{(N)}(r)\\ &\qquad = -2\sqrt{\lambda n\left(n+l+\frac{N}{2}-1\right)\left(n+l+\frac{N}{2}-2\right)}\, R_{n-1,l-1}^{(N)}(r).\end{aligned} \tag{3.16}$$

- $n_1 = 0, l_1 = -1$:

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} + \lambda r + \frac{l+N-2}{r}\right] R_{n,l}^{(N)}(r) = 2\sqrt{\lambda\left(n+l+\frac{N}{2}-1\right)}\, R_{n,l-1}^{(N)}(r). \tag{3.17}$$

- $n_1 = 0, l_1 = 1$:

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} - \lambda r - \frac{l}{r}\right] R_{n,l}^{(N)}(r) = -2\sqrt{\lambda\left(n+l+\frac{N}{2}\right)}\, R_{n,l+1}^{(N)}(r). \tag{3.18}$$

- $n_1 = 1, l_1 = 0$:

$$\left[r\left(\frac{\mathrm{d}}{\mathrm{d}r} - \lambda r\right) + (2n+l+N)\right] R_{n,l}^{(N)}(r) = 2\sqrt{(n+1)\left(n+l+\frac{N}{2}\right)}\, R_{n+1,l}^{(N)}(r). \tag{3.19}$$

- $n_1 = -1, l_1 = 0$:

$$\left[r\left(\frac{\mathrm{d}}{\mathrm{d}r} + \lambda r\right) - (2n+l)\right] R_{n,l}^{(N)}(r) = -2\sqrt{n\left(n+l+\frac{N}{2}-1\right)}\, R_{n-1,l}^{(N)}(r). \tag{3.20}$$

The formulae (3.13), (3.14), (3.17) and (3.15) correspond (up to minor misprints) to the formulae (16c), (16a), (16b) and (16d) of [6, p 4765], respectively, and generalize the ones obtained in [13].

4. Radial functions for the hydrogen atom

In this section we will provide a similar study for the N-dimensional hydrogen atom described by the Schrödinger equation

$$\left(-\Delta - \frac{1}{r}\right)\Psi = E\Psi \qquad \Delta = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \qquad r = \sqrt{\sum_{k=1}^{n} x_k^2}.$$

The solution is given by $\Psi = R_{nl}^{(N)}(r) Y_{lm}(\Omega_N)$, where the radial part $R_{nl}^{(N)}(r)$ is defined by [2, 14]

$$R_{nl}^{(N)}(r) = \mathcal{N}_{n,l}^{(N)} \left(\frac{r}{n+\frac{N}{2}-\frac{3}{2}}\right)^{l} \exp\left(-\frac{r}{2\left(n+\frac{N}{2}-\frac{3}{2}\right)}\right) L_{n-l-1}^{2l+N-2}\left(\frac{r}{n+\frac{N}{2}-\frac{3}{2}}\right). \tag{4.1}$$

Here $n = l+1, l+2, \ldots$ and $l = 0, 1, 2, \ldots$ are the quantum numbers, $N \geqslant 3$ is the dimension of the space, and the normalizing constant $\mathcal{N}_{n,l}^{(N)}$ is

$$\mathcal{N}_{n,l}^{(N)} = \sqrt{\frac{(n-l-1)!}{(n+l+N-3)!}} \frac{2}{\left(n+\frac{N-3}{2}\right)^2}. \tag{4.2}$$

As before, $Y_{lm}(\Omega_N)$ denotes the hyperspherical harmonics.

Here it is important to note that the Laguerre polynomials that appear in the expression of the radial functions are not the classical ones $L_n^{\alpha}(x)$ in the sense that the parameter α as well as the variable x depend on the degree of the polynomials, n. Nevertheless, the algebraic properties of the classical Laguerre polynomials (2.6)–(2.11) can be used for deriving the algebraic relations of the radial wavefunctions as we will show in this section. When the parameters of the classical polynomials depend on n, the polynomials are orthogonal with respect to a *variant weights* [15, 16, 29]. Using the theory of these variant classical polynomials the same recurrence relations can be derived as is shown in [29]. For more details on these *varying classical* polynomials we refer the reader to the aforementioned works [15, 16, 29].

4.1. Recurrence relations and ladder-type operators for the radial functions

We start by proving the following general theorem:

Theorem 4.1. *Let the functions* $R_{nl}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right]$, $R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\frac{N-3}{2}\right)r\right]$ *and* $R_{n+n_2,l+l_2}^{(N)}\left[\left(n+n_2+\frac{N-3}{2}\right)r\right]$ *be three different radial functions of the Nth hydrogen atom and* n_1, n_2 *and* l_1, l_2 *integers such that* $\min(n+n_1, n+n_2, l+l_1, l+l_2) \geqslant 0$. *Then, there exist non-vanishing polynomials in* r, A_0, A_1, *and* A_2, *such that*

$$A_0 R_{nl}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right] + A_1 R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\frac{N-3}{2}\right)r\right] + A_2 R_{n+n_2,l+l_2}^{(N)}\left[\left(n+n_2+\frac{N-3}{2}\right)r\right] = 0.$$

Proof. As in theorem 2.1 we will prove only the case when l_1, l_2 are non-negative integers since the $R_{nl}^{(N)}$, $R_{n+n_1,l\pm l_1}^{(N)}$, $R_{n+n_2,l\pm l_2}^{(N)}$ can be reduced to $R_{nl}^{(N)}$, $R_{n+n_1,l+l_1}^{(N)}$, $R_{n+n_2,l+l_2}^{(N)}$ by choosing $l = \min(l, l \pm l_1, l \pm l_2)$.

We start from equation (4.1)

$$L_{n-l-1}^{2l+N-2}\left(\frac{x}{n+\frac{N-3}{2}}\right) = \left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} \left(\frac{x}{n+\frac{N-3}{2}}\right)^{-l} \exp\left(\frac{x}{2\left(n+\frac{N-3}{2}\right)}\right) R_{nl}^{(N)}(x)$$

or equivalently, taking $x\big/\left(n+\frac{N-3}{2}\right)=r$, $\alpha=2l+N-2$, and $m=n-l-1$

$$L_m^\alpha(r)=\left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} r^{-l}\,\mathrm{e}^{r/2} R_{nl}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right]. \tag{4.3}$$

In a similar way

$$L_{m+n_1-l_1}^{\alpha+2l_1}(r)=\left(\mathcal{N}_{n+n_1,l+l_1}^{(N)}\right)^{-1} r^{-(l+l_1)}\,\mathrm{e}^{r/2} R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\frac{N-3}{2}\right)r\right] \tag{4.4}$$

$$L_{m+n_2-l_2}^{\alpha+2l_2}(r)=\left(\mathcal{N}_{n+n_2,l+l_2}^{(N)}\right)^{-1} r^{-(l+l_2)}\,\mathrm{e}^{r/2} R_{n+n_2,l+l_2}^{(N)}\left[\left(n+n_2+\frac{N-3}{2}\right)r\right]. \tag{4.5}$$

On the other hand

$$L_{m+n_1-l_1}^{\alpha+2l_1}(r)=\frac{\mathrm{d}^{2l_1}}{\mathrm{d}r^{2l_1}}L_{m+n_1+l_1}^{\alpha}(r) \qquad \text{and} \qquad L_{m+n_2-l_2}^{\alpha+2l_2}(r)=\frac{\mathrm{d}^{2l_2}}{\mathrm{d}r^{2l_2}}L_{m+n_2+l_2}^{\alpha}(r).$$

Now, using theorem 2.1 we can assure that there exist non-vanishing polynomials in r, A_0^*, A_1^*, and A_2^*, such that

$$A_0^*L_m^\alpha(r)+A_1^*\frac{\mathrm{d}^{2l_1}}{\mathrm{d}r^{2l_1}}L_{m+n_1+l_1}^\alpha(r)+A_2^*\frac{\mathrm{d}^{2l_2}}{\mathrm{d}r^{2l_2}}L_{m+n_2+l_2}^\alpha(r)=0$$

that is,

$$A_0^*L_m^\alpha(r)+A_1^*L_{m+n_1-l_1}^{\alpha+2l_1}(r)+A_2^*L_{m+n_2-l_2}^{\alpha+2l_2}(r)=0. \tag{4.6}$$

Replacing the expressions $L_m^\alpha(r)$, $L_{m+n_1-l_1}^{\alpha+2l_1}(r)$ and $L_{m+n_2-l_2}^{\alpha+2l_2}(r)$ by the right-hand sides of (4.3)–(4.5), respectively, in (4.6), we have

$$\begin{aligned}
&A_0^*\left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} R_{nl}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right]\\
&\qquad+A_1^*\left(\mathcal{N}_{n+n_1,l+l_1}^{(N)}\right)^{-1} r^{-l_1} R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\frac{N-3}{2}\right)r\right]\\
&\qquad+A_2^*\left(\mathcal{N}_{n+n_2,l+l_2}^{(N)}\right)^{-1} r^{-l_2} R_{n+n_2,l+l_2}^{(N)}\left[\left(n+n_2+\frac{N-3}{2}\right)r\right]=0
\end{aligned} \tag{4.7}$$

that proves the theorem with

$$A_0=A_0^*\left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} r^{l_1+l_2} \qquad A_1=A_1^*\left(\mathcal{N}_{n+n_1,l+l_1}^{(N)}\right)^{-1} r^{l_2} \qquad A_2=A_2^*\left(\mathcal{N}_{n+n_2,l+l_2}^{(N)}\right)^{-1} r^{l_1}.$$

□

Just as in the IHO case from a long list of interesting cases, we will list some examples of application of the last theorem.

- Putting $n_1=1, n_2=-1, l_1=l_2=0$ in (4.6), identifying the corresponding coefficients using the relation (2.11) of the Laguerre polynomials, and then using the relation (4.1) we find the expression

$$\begin{aligned}
&A_0R_{n-1,l}^{(N)}\left[\left(n+\frac{N}{2}-\frac{5}{2}\right)r\right]+A_1R_{n,l}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right]\\
&\qquad+A_2R_{n+1,l}^{(N)}\left[\left(n+\frac{N}{2}-\frac{1}{2}\right)r\right]=0
\end{aligned} \tag{4.8}$$

$$A_0=\sqrt{(n-l-1)(n+l+N-3)}\left(\frac{n+(N-5)/2}{n+(N-3)/2}\right)^2 \qquad A_1=-(2n+N-3-r)$$

$$A_2=\sqrt{(n-l)(n+l+N-2)}\left(\frac{n+(N-1)/2}{n+(N-3)/2}\right)^2.$$

- For $n_1 = n_2 = 0$ and $l_1 = -1, l_2 = 1$ we have

$$\begin{aligned}
&A_0 R^{(N)}_{n,l-1}(r) + A_1 R^{(N)}_{n,l}(r) + A_2 R^{(N)}_{n,l+1}(r) = 0 \\
&A_0 = (2l + N - 1)\sqrt{(n-l)(n+l+N-3)}r \\
&A_2 = (2l + N - 3)\sqrt{(n-l-1)(n+l+N-2)}r \\
&A_1 = (2l + N - 2)\left[(2n + N - 3)r - (2l+N-3)(2l+N-1)\left(n + \frac{N-3}{2}\right)\right].
\end{aligned} \tag{4.9}$$

- $n_1 = 0, n_2 = 1, l_1 = -1$ and $l_2 = 0$. In this case we obtain

$$\begin{aligned}
&A_0 R^{(N)}_{n,l-1}\left[\left(n + \frac{N-3}{2}\right)r\right] + A_1 R^{(N)}_{n,l}\left[\left(n + \frac{N-3}{2}\right)r\right] \\
&\qquad + A_2 R^{(N)}_{n+1,l}\left[\left(n + \frac{N}{2} - \frac{1}{2}\right)r\right] = 0 \\
&A_0 = r\sqrt{n+l+N-3} \qquad A_1 = \sqrt{n-l}[(2l+N-3)+r] \\
&A_2 = -(2l+N-3)\sqrt{n+l+N-2}\left(\frac{n+(N-1)/2}{n+(N-3)/2}\right)^2.
\end{aligned} \tag{4.10}$$

Other cases can be obtained in the same way.

Now we will state for the Nth hydrogen atom the so-called ladder-type operators.

Theorem 4.2. *Let $R^{(N)}_{nl}\left[\left(n + \frac{N-3}{2}\right)r\right]$ and $R^{(N)}_{n+n_1,l+l_1}\left[\left(n + n_1 + \frac{N-3}{2}\right)r\right]$ be two different radial functions of the Nth hydrogen atom and $\frac{\mathrm{d}}{\mathrm{d}r} R^{(N)}_{nl}\left[\left(n + \frac{N-3}{2}\right)r\right]$ the first derivative with respect to r, where n_1 and l_1 are integers such that* $\min(n+n_1, l+l_1) \geqslant 0$, $(n_1)^2 + (l_1)^2 \neq 0$. *Then, there exist non-vanishing polynomials in r, A_0, A_1 and A_2, such that*

$$\begin{aligned}
&A_0 R^{(N)}_{nl}\left[\left(n + \frac{N-3}{2}\right)r\right] + A_1 \frac{\mathrm{d}}{\mathrm{d}r} R^{(N)}_{nl}\left[\left(n + \frac{N-3}{2}\right)r\right] \\
&\qquad + A_2 R^{(N)}_{n+n_1,l+l_1}\left[\left(n + n_1 + \frac{N-3}{2}\right)r\right] = 0.
\end{aligned} \tag{4.11}$$

Proof. The proof is similar to that of theorem 3.2 so we will present here only a sketch of it. In fact, using (4.1), we have

$$L^{\alpha}_m(z) = \left(\mathcal{N}^{(N)}_{n,l}\right)^{-1} z^{-l}\, \mathrm{e}^{\frac{z}{2}} R^{(N)}_{nl}\left[\left(n + \frac{N-3}{2}\right)z\right]$$

$$\text{where} \qquad z = \frac{r}{n + \frac{N-3}{2}} \qquad \alpha = 2l + N - 2 \qquad m = n - l - 1.$$

Taking the derivative of the above formula with respect to z, using the identity (2.6) for the Laguerre polynomials and the theorem 2.1 we can guarantee the existence of non-vanishing polynomials B_0, B_1, and B_2 such that

$$B_0 L^{\alpha}_m(z) + B_1 L^{\alpha+1}_{m-1}(z) + B_2 L^{\alpha+2l_1}_{m+n_1-l_1}(z) = 0. \tag{4.12}$$

Now, using relation (4.1), we obtain

$$\begin{aligned}
&r^{l_1}\left(B_1 \frac{\mathrm{d}}{\mathrm{d}r} - B_1 \frac{l}{r} + \frac{1}{2}B_1 - B_0\right) R^{(N)}_{nl}\left[\left(n + \frac{N-3}{2}\right)r\right] \\
&\qquad = B_2 \frac{\mathcal{N}^{(N)}_{n,l}}{\mathcal{N}^{(N)}_{n+n_1,l+l_1}} R^{(N)}_{nl}\left[\left(n + n_1 + \frac{N-3}{2}\right)r\right]
\end{aligned} \tag{4.13}$$

from where the results immediately follow. □

Again, from the above theorem it is easy to obtain several relations for the radial wavefunctions of the Nth hydrogen atom and, in particular, the ladder operators in n and l, respectively.

- $n_1 = 0, l_1 = 1$: we start by substituting $n_1 = 0$ and $l_1 = 1$ in (4.12)

$$B_0 L_m^{\alpha}(z) + B_1 L_{m-1}^{\alpha+1}(z) + B_2 L_{m-1}^{\alpha+2}(z) = 0 \qquad \text{where} \qquad m = n - l - 1, \alpha = 2l + N - 2.$$

Next, we use (2.8) and (2.6), respectively, to 'eliminate' $L_m^{\alpha}(z)$ and $L_{m-1}^{\alpha+2}(z)$. This yields,

$$B_0 L_m^{\alpha+1}(z) + (B_1 - B_0) L_{m-1}^{\alpha+1}(z) - B_2 \frac{\mathrm{d}}{\mathrm{d}z} L_m^{\alpha+1}(z) = 0.$$

Then, comparing the resulting equation with expression (2.7), we find $B_0 = m$, $B_2 = r$, $B_1 - B_0 = -(m + \alpha + 1)$, i.e. $B_0 = n - l - 1$, $B_1 = -(2l + N - 1)$, and $B_2 = r$, then (4.13), after the change $\left(n + \frac{N-3}{2}\right) r \to r$, leads to the following ladder-type relation

$$\left[\left(l + \frac{N-1}{2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l}{r}\right) + \frac{1}{2}\right] R_{nl}^{(N)}(r) = -\frac{1}{2}\sqrt{1 - \left(\frac{l + (N-1)/2}{n + (N-3)/2}\right)^2}\, R_{n,l+1}^{(N)}(r). \tag{4.14}$$

- $n_1 = 0, l_1 = -1$: substituting these values in (4.12) we obtain

$$B_0 L_m^{\alpha}(z) + B_1 L_{m-1}^{\alpha+1}(z) + B_2 L_{m+1}^{\alpha-2}(z) = 0 \qquad \text{where} \qquad m = n - l - 1, \alpha = 2l + N - 2.$$

Next, we use (2.8) and (2.10) twice, which leads to

$$B_2 L_{m+1}^{\alpha}(z) + \left(B_0 - B_1 \frac{m}{z} - 2B_2\right) L_m^{\alpha}(z) - \left(B_1 \frac{m+\alpha}{z} + B_2\right) L_{m-1}^{\alpha}(z) = 0.$$

Comparing this expression with (2.11) we find $B_2 = m + 1$, $B_0 - B_1 m/z - 2B_2 = -2m - \alpha - 1 + z$ and $B_1(m+\alpha)/z + B_2 = m + \alpha$, i.e. $B_0 = [(m+\alpha)z - \alpha(\alpha-1)]/(m+\alpha)$, $B_1 = (\alpha - 1)z/(m+\alpha)$. Substituting the functions B_0, B_1, and B_2 in (4.13) we finally obtain

$$\left[\left(l + \frac{N}{2} - \frac{3}{2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l + N - 2}{r}\right) - \frac{1}{2}\right] R_{nl}^{(N)}(r) = \frac{1}{2}\sqrt{1 - \left(\frac{l + (N-3)/2}{n + (N-3)/2}\right)^2}\, R_{n,l-1}^{(N)}(r). \tag{4.15}$$

In the same way we find

- $n_1 = -1, l_1 = -1$:

$$\left[(2l + N - 3 + r)\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{2}\right) + \frac{(2l + N - 3)(l + N - 2)}{r} - (n-1)\right] \times R_{nl}^{(N)}\left[\left(n + \frac{N-3}{2}\right) r\right] = \sqrt{(n + l + N - 3)(n + l + N - 4)} \times \left(\frac{n + (N-5)/2}{n + (N-3)/2}\right)^2 R_{n-1,l-1}^{(N)}\left[\left(n + \frac{N-5}{2}\right) r\right]. \tag{4.16}$$

- $n_1 = 1, l_1 = 1$:

$$\left[(2l + N - 1 + r)\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l}{r} - \frac{1}{2}\right) + (n + l + N - 2)\right] R_{nl}^{(N)}\left[\left(n + \frac{N-3}{2}\right) r\right] = -\sqrt{(n + l + N - 1)(n + l + N - 2)}\left(\frac{n + (N-1)/2}{n + (N-3)/2}\right)^2 \times R_{n+1,l+1}^{(N)}\left[\left(n + \frac{N-1}{2}\right) r\right]. \tag{4.17}$$

- $n_1 = -1, l_1 = 1$:

$$\left[(r-(2l+N-1))\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l}{r}+\frac{1}{2}\right)-(n-l-1)\right]R_{nl}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right]$$
$$=\sqrt{(n-l-1)(n-l-2)}\left(\frac{n+(N-5)/2}{n+(N-3)/2}\right)^2 R_{n-1,l+1}^{(N)}\left[\left(n+\frac{N-5}{2}\right)r\right]. \tag{4.18}$$

- $n_1 = 1, l_1 = -1$:

$$\left[(2l+N-3-r)\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{l+N-2}{r}-\frac{1}{2}\right)-(n-l)\right]R_{nl}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right]$$
$$=\sqrt{(n-l)(n-l-1)}\left(\frac{n+(N-1)/2}{n+(N-3)/2}\right)^2 R_{n+1,l-1}^{(N)}\left[\left(n+\frac{N-1}{2}\right)r\right]. \tag{4.19}$$

- $n_1 = -1, l_1 = 0$:

$$\left[r\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{2\left(n+\frac{N-3}{2}\right)}\right)-(n-1)\right]R_{nl}^{(N)}(r)$$
$$= -\sqrt{(n-l-1)(n+l+N-3)}\left(\frac{n+(N-5)/2}{n+(N-3)/2}\right)^2$$
$$\times R_{n-1,l}^{(N)}\left[\left(\frac{n+(N-5)/2}{n+(N-3)/2}\right)r\right]. \tag{4.20}$$

- $n_1 = 1, l_1 = 0$:

$$\left[r\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{1}{2\,(n+(N-3)/2)}\right)+(n+N-2)\right]R_{nl}^{(N)}(r)$$
$$=\sqrt{(n-l)(n+l+N-2)}\left(\frac{n+(N-1)/2}{n+(N-3)/2}\right)^2 R_{n+1,l}^{(N)}\left[\left(\frac{n+(N-1)/2}{n+(N-3)/2}\right)r\right]. \tag{4.21}$$

Finally, let us point out that, as we already mentioned in the introduction, some of the above relations have also been obtained in other papers, e.g., the expressions (4.15), (4.14), (4.20) and (4.21) correspond to the formulae (31), (32), (34) and (33) of [6, p 4767], respectively; our relation (4.14) corresponds to (4.8) in [14, p 1070], and generalize equations (40) of [21, p 182] and equation (1) of [22, p 472]; relation (4.15) is equivalent to (4.9) in [14, p 1070], and generalize equation (43) of [21, p 182] and equation (2) of [22, p 472]; equations (4.20) and (4.21) are equivalent to equation (8) of [18, p 4260]; the relations (3.16–3.19) of [25, p 89] are particular cases of the relations (4.15), (4.14), (4.21), (4.20), respectively.

5. Concluding remarks

In this paper we present a very simple and constructive approach for finding recurrence relations for the radial wavefunctions of two very important systems: the isotropic harmonic oscillator and the non-relativistic hydrogen atom in N dimensions. Some of the relations were discovered by different methods (sometimes using very cumbersome calculations) and play a fundamental role in several theories, e.g., the ladder relations are closly related to the factorization method of Infeld and Hull for Sturm–Liouville problems, and some of them can be used for numerical computations of these functions. Furthermore, the method presented here is also valid for other quantum systems such as, for instance, the Morse problem [21] and the relativistic hydrogen atom [25], since the corresponding wavefunctions are proportional to the Laguerre polynomials. Obviously, this method for finding recurrence relations can be extended to any quantum system whose (radial) wavefunction is proportional to hypergeometric-type functions (see, e.g., [5]).

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